Admissions Testing Service

STEP Solutions 2013

Mathematics

STEP 9465/9470/9475

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Test

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STEP Mathematics (9465, 9470, 9475)

Q1 This question is all about using substitutions to simplify the working required to solve various increasingly complicated looking equations. To begin with, you are led gently by the hand in (i), where the initial substitution has been given directly to you. You are also reminded that y must be non-negative, since $\sqrt{x}$ denotes the *positive* square-root of x (positive unless x is 0, of course). The result is obviously a quadratic equation, $y^2 + 3y - \frac{1}{2} = 0$, and is solvable by (for instance) use of the quadratic formula. However, only one of the apparent solutions, $y = \dfrac{-3 \pm \sqrt{11}}{2}$, is positive, so the other is rejected and we proceed to find that $x = \left(\dfrac{\sqrt{11}-3}{2}\right)^2 = 5 - \frac{3}{2}\sqrt{11}$.

In (ii) (a), the approach used in (i) should lead you to consider the substitution $y = \sqrt{x+2}$, which gives the quadratic equation $y^2 + 10y - 24 = 0$. This, in turn, yields $y = \sqrt{x+2} = -12$ or 2. Again, this must be non-negative, so we find that $\sqrt{x+2} = 2$ and $x = 2$.

In (ii) (b), it should not now be too great a leap of faith to set $y = \sqrt{2x^2 - 8x - 3}$ which, with a bit of modest tinkering, yields up $y^2 + 2y - 15 = 0 \Rightarrow y = \sqrt{2x^2 - 8x - 3} = -5,\ 3$. Again, $y \geq 0$, so $\sqrt{2x^2 - 8x - 3} = 3 \Rightarrow$ (with cancelling) $x^2 - 4x - 6 = 0 \Rightarrow x = 2 \pm \sqrt{10} \Rightarrow x^2 = 14 \pm 4\sqrt{10}$. The final step here is to check that both apparent solutions work in the original equation, since the squaring process usually creates invalid solutions. [Note that it is actually quite easy to see that both solutions are indeed valid, but this still needs to be shown, or otherwise explained. Longer methods involving much squaring usually generated four solutions, two of which were *not* valid.]

Q2 The "leading actor" throughout this question is the *integer-part* function, $\lfloor x \rfloor$, often referred to as the *floor-function*. Purely as an aside, future STEP candidates may find it beneficial to play around with such strange, possibly artificial, kinds of functions as part of their preparation because, although they clearly go beyond the scope of standard syllabuses at this level, they *are* within reach and require little more than a willingness to be challenged. [Note that some care needs to be taken when exploring such things. In the case of this *floor-function*, at least a couple of function plotting software programs that recognise the "INT" function do so incorrectly for $x < 0$: for instance, interpreting INT(–2.7) as –2 rather than –3.]

The key elements of the sketch in (i) are as follows. The jump in the value of $\lfloor x \rfloor$ whenever x hits an integer value means that the graph is composed of lots of "unit" segments, the LH end of which is included but not the RH end. The usual convention for signalling these properties is that the LH endpoint has a filled-in dot while the RH endpoint has an open dot. Then, in-between integer values of x, each segment of the curve is of the form $\dfrac{n}{x}$ and thus appears to be a portion of a reciprocal curve.

The purpose of parts (ii) and (iii) is to see if you can use your graph to decide how to solve some otherwise fairly simple equations: the key is to have a clear idea as to where the various portions of the graph exist. The analysis looks complicated, but candidates were actually only required to pick the appropriate n's and write down the relevant answers (so the working only needed to reflect what was going on "inside one's head"). Note that, for $n \le x < n+1$, $\lfloor x \rfloor = n$ so $f(x) = \frac{n}{x}$. Also, $\frac{n}{n+1} < f(x) \le 1$ for $x > 0$, and $f(x) \ge 1$ for $x < 0$, so $f(x) = \frac{7}{12}$ only in [1, 2), yielding the equation $\frac{1}{x} = \frac{7}{12}$ in (ii) $\Rightarrow x = \frac{12}{7}$.

Similarly, $\frac{n}{n+1} > \frac{17}{24} \Rightarrow 24n > 17n + 17 \Rightarrow n > 2\frac{3}{7}$, i.e. $n \ge 3$; so $f(x) = \frac{17}{24}$ only in [1, 2) and [2, 3). In [1, 2), $f(x) = \frac{1}{x} = \frac{17}{24} \Rightarrow x = \frac{24}{17}$ and in [2, 3), $f(x) = \frac{2}{x} = \frac{17}{24} \Rightarrow x = \frac{48}{17}$. Next, for $x < 0$, $1 \le f(x) < \frac{n}{n+1}$, and $\frac{-n}{-n-1} < \frac{4}{3} \Rightarrow -4n - 4 > -3n \Rightarrow n < -4$; so $f(x) = \frac{4}{3}$ only in [–4, –3), [–3, –2), [–2, –1) and [–1, 0). However, since $f(-3) = 1$ there is no solution in [–4, –3). Otherwise, in [–3, –2), $f(x) = \frac{-3}{x} = \frac{4}{3} \Rightarrow x = -\frac{9}{4}$; in [–2, –1), $f(x) = \frac{-2}{x} = \frac{4}{3} \Rightarrow x = -\frac{3}{2}$; and in [–1, 0), $f(x) = \frac{-1}{x} = \frac{4}{3} \Rightarrow x = -\frac{3}{4}$.

For (iii), $\frac{n}{n+1} > \frac{9}{10}$ for $n > 9$ so $f(x_{\max}) = \frac{9}{10}$ in [8, 9) and $f(x) = \frac{8}{x} = \frac{9}{10} \Rightarrow x = \frac{80}{9}$.

Only the very last part required any great depth of insight, and the ability to hold one's nerve. The equation $f(x) = c$ has exactly n roots when the horizontal line $y = c$ cuts the curve that number of times. That is …

… when $x > 0$: $\frac{n}{n+1} < c \le \frac{n+1}{n+2}$; … when $x < 0$: $\frac{n+1}{n} \le c < \frac{n}{n-1}$, $n \ge 2$; … and $c \ge 2$ for $n = 1$.

Q3 This vector question is tied up with the geometric understanding that, for distinct points with position vectors **x** and **y**, the point with p.v. $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ cuts XY in the ratio $(1-\lambda):\lambda$ (though it is important to realise that this point is only between X and Y if $0 < \lambda < 1$). Part (i) tests (algebraically) the property of *commutativity* (whether the composition yields different results if the order of the application of the operation is changed):

$$X*Y = Y*X \quad \Leftrightarrow \lambda\mathbf{x} + (1-\lambda)\mathbf{y} = \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \Leftrightarrow (2\lambda - 1)(\mathbf{x} - \mathbf{y}) = \mathbf{0} \Leftrightarrow (\text{since } \mathbf{x} \ne \mathbf{y})\ \lambda = \tfrac{1}{2}.$$

Part (ii) then explores the property of *associativity* (whether the outcome is changed when the order of the elements involved in two successive operations remains the same but the pairings within those successive operations is different). Here we have

$$(X*Y)*Z = \lambda(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) + (1-\lambda)\mathbf{z} = \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$$

and $$X*(Y*Z) = \lambda\mathbf{x} + (1-\lambda)[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}] = \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$$

Thus, $(X*Y)*Z - X*(Y*Z) = \lambda(1-\lambda)(\mathbf{x} - \mathbf{z})$ and the two are distinct provided $\lambda \ne 0, 1$ or $X \ne Z$.

Part (iii) now explores a version of the property of *distributivity* (although usually referring to two distinct operations): $(X*Y)*Z = \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$, and

$$(X*Z)*(Y*Z) = [\lambda\mathbf{x} + (1-\lambda)\mathbf{z}]*[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}] = \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{z} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$$

$$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}, \text{ and the two are always equal.}$$

Next, $X*(Y*Z) = \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$, and

$$(X*Y)*(X*Z) = [\lambda\mathbf{x} + (1-\lambda)\mathbf{y}]*[\lambda\mathbf{x} + (1-\lambda)\mathbf{z}]$$
$$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + \lambda(1-\lambda)\mathbf{x} + (1-\lambda)^2\mathbf{z}$$
$$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}\,.$$

Hence $X*(Y*Z) = (X*Y)*(X*Z)$ also.

In (iv), you will notice that the condition $0 < \lambda < 1$ comes into play, so that P_1 cuts XY *internally* in the ratio $(1-\lambda)$**:**λ. Following this process through a couple more steps shows us that P_n cuts XY in the ratio $(1-\lambda^n)$**:**λ^n, which is easily established inductively.

Q4 The first part to this question involved two integrals which can readily be integrated by "recognition", upon spotting that

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\mathrm{f}(\tan x)\big) = \mathrm{f}'(\tan x)\times\sec^2 x \text{ and } \frac{\mathrm{d}}{\mathrm{d}x}\big(\mathrm{f}(\sec x)\big) = \mathrm{f}'(\sec x)\times\sec x\tan x\,.$$

(They can, of course be integrated using suitable substitutions, etc.) Thus, we have

$$\int \tan^n x.\sec^2 x\ \mathrm{d}x = \left[\frac{1}{n+1}\tan^{n+1} x\right] = \frac{1}{n+1}$$

and $\int \sec^n x.\tan x\ \mathrm{d}x = \int \sec^{n-1} x.\sec x\tan x\ \mathrm{d}x = \left[\frac{1}{n}\sec^n x\right] = \dfrac{\left(\sqrt{2}\right)^n - 1}{n}$.

The two integrals in part (ii) can be approached in many different ways – the examiners worked out more than 25 slightly different approaches, depending upon how, and when, one used the identity $\sec^2 x = 1 + \tan^2 x$, how one split the "parts" in the process of "integrating by parts", and even whether one approached the various secondary integrals that arose as a function of $\sec x$ or $\tan x$. Only one of these approaches appears below for each of these two integrals..

$$\int_0^{\pi/4} x\sec^4 x\tan x\,\mathrm{d}x = \left[x.\frac{\sec^4 x}{4}\right]_0^{\pi/4} - \int_0^{\pi/4}\frac{\sec^4 x}{4}\mathrm{d}x \ \ (\text{by } parts) = \frac{\pi}{4} - \frac{1}{4}J\,, \text{ where } J = \int_0^{\pi/4}\sec^4 x\ \mathrm{d}x.$$

Then, $J = \int_0^{\pi/4}\sec^2 x\ \mathrm{d}x + \int_0^{\pi/4}\sec^2 x\tan^2 x\ \mathrm{d}x = \left[\tan x + \tfrac{1}{3}\tan^3 x\right] = \dfrac{4}{3}$, and our integral is $\dfrac{\pi}{4} - \dfrac{1}{3}$.

Next, $\int x^2\left(\sec^2 x.\tan x\right)\mathrm{d}x = \left[x^2.\tfrac{1}{2}\tan^2 x\right] - \int 2x.\tfrac{1}{2}\tan^2 x\,\mathrm{d}x$ (by *parts*) $= \dfrac{\pi^2}{32} - \int x\left(\sec^2 x - 1\right)$

$$= \frac{\pi^2}{32} - K + \int x\,\mathrm{d}x, \text{ where } K = \int_0^{\pi/4} x\sec^2 x\ \mathrm{d}x.$$

Then, $K = [x.\tan x] - \int\tan x\ \mathrm{d}x = x\tan x - \ln(\sec x) = \dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$, so that this last integral is

$$\frac{\pi^2}{32} - \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) + \frac{\pi^2}{32} \text{ or } \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\ln 2\,.$$

Q5 This question simply explores the different possibilities that arise when considering curves of a particular quadratic form. In (i), with a zero product term, we have equal amounts of x^2 and y^2, and this is symptomatic of a circle's equation: $x^2+3x+y^2+y=0 \Leftrightarrow \left(x+\frac{3}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=\left(\frac{1}{2}\sqrt{10}\right)^2$, which is a circle with centre $\left(-\frac{3}{2},-\frac{1}{2}\right)$ and radius $\frac{1}{2}\sqrt{10}$. This circle passes through the points (0, 0), (0, –1) & (–3, 0).

In (ii), with $k = \frac{10}{3}$, we have $(3x+y)(x+3y+3)=0$. This factorisation tells us that we have the line-pair $y = -3x$ & $x + 3y = -3$; the first line passing through the origin with negative gradient, while the second cuts the coordinate axes at (0, –1) and (–3, 0).

Part (iii) is the genuinely tough part of the question, but help is given to point you in the right direction. When $k = 2$, we have $(x+y)^2+3x+y=0$, and using $\theta = 45^\circ$ in the given substitution gives $x+y=X\sqrt{2}$ and $y-x=Y\sqrt{2} \Rightarrow x=\dfrac{X-Y}{\sqrt{2}}$ and $y=\dfrac{X+Y}{\sqrt{2}}$. Then

$$(x+y)^2+3x+y=0 \text{ becomes } 2X^2+\frac{3X-3Y}{\sqrt{2}}+\frac{X+Y}{\sqrt{2}}=0 \Rightarrow 2X^2+2\sqrt{2}X=Y\sqrt{2} \text{ or}$$

$\left(\sqrt{2}X+1\right)^2-1=Y\sqrt{2}$. This is now in what should be a familiar form for a parabola, with axis of symmetry $X=\dfrac{-1}{\sqrt{2}} \Rightarrow$ (substituting back) $\dfrac{x+y}{\sqrt{2}}=\dfrac{-1}{\sqrt{2}}$ i.e. $x+y=-1$. For the sketch, we must rotate the standard parabola through 45° *anticlockwise* about O in order to get the original parabola $x^2+2xy+y^2+3x+y=0$.

For those who have encountered such things, all three curves here are examples of *conic sections*.

Q6 It should be pretty clear that this question is all about binomial coefficients. The opening result – the well-known *Pascal Triangle* formula for generating one row's entries from those of the previous row – is reasonably standard and can be established in any one of several ways. The one intended here was as follows: the coefficient of x^r in $(1 + x)^{n+1}$ is $\binom{n+1}{r}$, and this is obtained from $(1 + x)(1 + x)^n$, where the coefficient of x^r comes from

$$(1+x)\left(\;\ldots\ldots+\binom{n}{r-1}x^{r-1}+\binom{n}{r}x^r+\ldots\ldots\;\right)$$

and the required result follows.

In the next stage, for n even, write $n = 2m$ so that

$$B_{2m}+B_{2m+1}=\binom{2m}{0}+\binom{2m-1}{1}+\binom{2m-2}{2}+\ldots\ldots+\binom{m+1}{m-1}+\binom{m}{m}$$
$$+\binom{2m+1}{0}+\binom{2m}{1}+\binom{2m-1}{2}+\binom{2m-2}{3}+\ldots\ldots+\binom{m+1}{m}.$$

and, pairing these terms suitably, this is

$$\binom{2m+1}{0}+\left[\binom{2m}{0}+\binom{2m}{1}\right]+\left[\binom{2m-1}{1}+\binom{2m-1}{2}\right]+\ldots\ldots+\left[\binom{m+1}{m-1}+\binom{m+1}{m}\right]+\binom{m}{m}.$$

Now, using the opening result, and the fact that $\binom{2m+1}{0}=\binom{2m+2}{0}=\binom{m}{m}=\binom{m+1}{m+1}=1$, we have

$$\binom{2m+2}{0} + \left[\binom{2m+1}{1}\right] + \left[\binom{2m}{2}\right] + \ldots\ldots + \left[\binom{m+2}{m}\right] + \binom{m+1}{m+1},$$

and this is just $\sum_{j=0}^{m+1}\binom{2(m+1)-j}{j} = B_{2m+2}$, as required.

In the case n odd, write $n = 2m + 1$, so that

$$B_{2m+1} + B_{2m+2} = \binom{2m+1}{0} + \binom{2m}{1} + \binom{2m-1}{2} + \ldots\ldots + \binom{m+2}{m-1} + \binom{m+1}{m}$$
$$+ \binom{2m+2}{0} + \binom{2m+1}{1} + \binom{2m}{2} + \binom{2m-1}{3} + \ldots\ldots + \binom{m+2}{m} + \binom{m+1}{m+1}$$

which gives, upon pairing terms suitably,

$$\binom{2m+2}{0} + \left[\binom{2m+1}{0} + \binom{2m+1}{1}\right] + \left[\binom{2m}{1} + \binom{2m}{2}\right] + \ldots\ldots + \left[\binom{m+2}{m-1} + \binom{m+2}{m}\right] + \left[\binom{m+1}{m} + \binom{m+1}{m+1}\right]$$
$$= \binom{2m+3}{0} + \left[\binom{2m+2}{1}\right] + \left[\binom{2m+1}{2}\right] + \ldots\ldots + \left[\binom{m+3}{m}\right] + \binom{m+2}{m+1}$$

using the opening result and the fact that $\binom{2m+2}{0} = \binom{2m+3}{0} = 1$

$$= \sum_{j=0}^{m+1}\binom{2(m+1)+1-j}{j} = B_{2m+3}.$$

To complete an inductive proof, we must also check that the starting terms match up properly, but this is fairly straightforward. We conclude that, since $B_0 = F_1$, $B_1 = F_2$ and B_n & F_n satisfy the same recurrence relation, we must have $B_n = F_{n+1}$ for all n.

Q7 In (i), there is a generous tip given to help you on your way with this question. Starting from $y = ux$ we have $\frac{dy}{dx} = u + x\frac{du}{dx}$, so that the given differential equation becomes $u + x\frac{du}{dx} = \frac{1}{u} + u$ or $\int u\,du = \int \frac{1}{x}\,dx$ upon separation of variables. You are now in much more familiar territory and may proceed in the standard way: $\frac{1}{2}u^2 = \frac{y^2}{2x^2} = \ln x + C \Rightarrow y^2 = x^2(2\ln x + 2C)$. Using the given conditions $x = 1$, $y = 2$ to determine $C = 2$ then gives the required answer $y = x\sqrt{2\ln x + 4}$. However, there is one small detail still required, namely to justify the taking of the *positive* square-root, which follows from the fact that $y > 0$ when $x = 1$. (Note that you were given $x > e^{-2}$, for the validity of the square-rooting to stand, so it is not necessary to justify this. However, it should serve as a hint that a similar justification may be required in the later parts of the question.)

In (ii), either of the substitutions $y = ux$ or $y = ux^2$ could be used to solve this second differential equation. In each case, the method then follows that of part (i)'s solution very closely indeed; separating variables, integrating, eliminating u and substituting in the condition $x = 1$, $y = 2$ to evaluate the arbitrary constant. The final steps require a justifying of the taking of the positive square-root and a statement of the appropriate condition on x in order to render the square-rooting a valid thing to do. The answer is $y = x\sqrt{5x^2 - 1}$ for $x > \frac{1}{\sqrt{5}}$.

In (iii), only the substitution $y = ux^2$ can be used to get a variable-separable differential equation, which boils down to $\int u\,\mathrm{d}u = \int \frac{1}{x^2}\,\mathrm{d}x \Rightarrow \frac{1}{2}u^2 = \frac{-1}{x} + D$. Using $x = 1$, $y = 2$ ($u = 2$) to evaluate the constant D leads to the answer $y = x\sqrt{6x^2 - 2x}$ for $x > \frac{1}{3}$.

Q8 This question is all about composition of functions and their associated domains and ranges. Whilst being essentially a very simple question, there is a lot of scope for minor oversights. One particular pitfall is to think that $\sqrt{x^2} = x$, when it is actually $|\ x\ |$. Also, when considering the composite function fg, it is essential that the domain of g (the function that "acts" first on x) is chosen so that the output values from it are suitable input values for f. You should check that this is so for the four composites required in part (i).

In (ii), the functions fg and gf *look* the same (both are $|\ x\ |$) but their domains and ranges are different: fg has domain $\mathbb{R}$ and range $y \geq 0$, while the second has domain $|\ x\ | \geq 1$ and range $y \geq 1$.

In (iii), the essentials of the graph of h are: it starts from (1, 1) and increases. Since $\sqrt{x^2 - 1}$ is just (x – a tiny bit) after a while, the graph of h approaches $y = 2x$ from below. Using similar reasoning, the graph of k for $x \geq 1$, also starts at (1, 1) but decreases asymptotically to zero. (It is well worth noting that $x - \sqrt{x^2 - 1}$ is the reciprocal of $x + \sqrt{x^2 - 1}$, since their product is 1.) However, this second graph has a second branch for $x \leq -1$, which is easily seen to be a rotation about O (through 180°) of the single branch of h, this time approaching $y = 2x$ from above. Finally, note that the domain of kh is $x \geq 1$, and since the range of h is $y \geq 1$, the range of kh is $0 < y \leq 1$.

Q9 This question incorporates the topics of collisions and projectiles, each of which consists of several well-known and oft quoted results. However, much of the algebraic processing can be shortcut by a few insightful observations. To begin with, if the two particles start together at ground level and also meet at their highest points, then they must have the same vertical components of velocity. Thus $u\sin\alpha = v\sin\beta$. Then, if they both return to their respective points of projection, the collision must have ensured that they both left the collision with the same horizontal velocity as when they arrived; giving, by *Conservation of Linear Momentum*, that $mu\cos\alpha = Mv\cos\beta$. Dividing these two results gives the required answer, $m\cot\alpha = M\cot\beta$.

The collision occurs when $t = \dfrac{u\sin\alpha}{g} = \dfrac{v\sin\beta}{g}$ (from the standard constant-acceleration formulae) and at the point when A has travelled a distance $b = \dfrac{u^2\sin\alpha\cos\alpha}{g}$ and B has travelled a distance $\dfrac{v^2\sin\beta\cos\beta}{g} = \dfrac{1}{g}(v\sin\beta)(v\cos\beta)$, the sum of these two distances being denoted d. Substituting for the brackets using the two initial results then gives $b = \dfrac{Md}{m+M}$, as required. Moreover, the height of the two particles at the collision is given by $y = \dfrac{u^2\sin^2\alpha}{2g}$, so that

$$h = \frac{1}{2}\times\frac{u^2\sin\alpha\cos\alpha}{g}\times\frac{\sin\alpha}{\cos\alpha} = \tfrac{1}{2}b\tan\alpha\,.$$

Q10 This question makes more obvious use of the standard results for collisions, but also ties them up with *Newton's 2nd Law* (N2L) of motion and, implicitly, the *Friction Law* and resolution of forces (in the simplest possible form). Thus, if R is the normal contact reaction force of floor on puck, F the frictional resistance between floor and puck, we have (in very quick order) the results $R = mg$, $F = \mu R = \mu mg$ and, by N2L, $\mu mg = -ma$, where a is the puck's acceleration. The constant-acceleration formula $v^2 = u^2 - 2as$ then gives ${w_{i+1}}^2 = {v_i}^2 - 2\mu gd$ (where v_i is the speed of the puck when leaving the i-th barrier, for $i = 0, 1, 2, \ldots$, and w_i is its speed when arriving at the i-th barrier, for $i = 1, 2, 3, \ldots$). Also, *Newton's (Experimental) Law of Restitution* (NEL or NLR) gives $v_{i+1} = r.w_{i+1}$, from which it follows that ${v_{i+1}}^2 = r^2{v_i}^2 - 2r^2\mu gd$, as required.

Iterating with this result, starting with ${v_1}^2 = r^2v^2 - 2r^2\mu gd$, then leads to the general result ${v_n}^2 = r^{2n}v^2 - 2r^2\mu gd\{1 + r^2 + r^4 + \ldots + r^{2n-2}\}$. The large bracket is the sum-to-n-terms of a GP, namely $\dfrac{1-r^{2n}}{1-r^2}$, and you simply need to set $v_n = 0$ and tidy up in order to obtain the result

$$\frac{v^2}{2\mu gd}r^{2n} = \frac{(1-r^{2n})r^2}{1-r^2}.$$

Replacing $\dfrac{v^2}{2\mu gd}$ by k and re-writing the expression for r^{2n} $\left(= \dfrac{r^2}{r^2 + k(1-r^2)}\right)$, we simply have to take logs and solve for n to get $n = \dfrac{\ln\left(\dfrac{r^2}{r^2+k(1-r^2)}\right)}{2\ln r}$. Setting $r = \mathrm{e}^{-1}$ in this result, and tidying up, then gives $n = \tfrac{1}{2}\ln(1 + k(\mathrm{e}^2 - 1))$.

When $r = 1$, the distance travelled is just nd, and we have $v^2 = 2\mu gnd$ and $n = \dfrac{v^2}{2\mu gd} = k$.

Q11 Since we are told that $\alpha + \beta < \tfrac{1}{2}\pi$, the two tensions $T\sin\beta$ and $T\cos\beta$ are effectively components of a notional force ($\boldsymbol{T}$), inclined slightly to the right of the normal contact reaction force R, say. Hence, if there is motion, it will take place to the right. Then, resolving vertically and horizontally for the block, calling the frictional force (acting to the left) F, we have

$$R + T\sin\beta\cos\alpha + T\cos\beta\sin\alpha = W \text{ and } F + T\sin\beta\sin\alpha = T\cos\beta\cos\alpha$$

or, using trig. identities, $W = R + T\sin(\alpha + \beta)$ and $F = T\cos(\alpha + \beta)$. Since $W > T\sin(\alpha + \beta)$, it follows that $R > 0$ so the block does not rise. Otherwise, $F \le \mu R$ and using $\mu = \tan\lambda$ for equilibrium, we have $T\cos(\alpha + \beta) \le \tan\lambda\,\{W - T\sin(\alpha+\beta)\}$ i.e. $W\tan\lambda \ge T\tan\lambda\sin(\alpha + \beta) + T\cos(\alpha + \beta)$ $\Rightarrow$ $W\sin\lambda \ge T\sin\lambda\sin(\alpha + \beta) + T\cos(\alpha + \beta)\cos\lambda = T\cos(\alpha + \beta - \lambda)$.

In the next part, $W = T\tan\tfrac{1}{2}(\alpha+\beta)$ $\Rightarrow$ $R = T\sin\tfrac{1}{2}(\alpha+\beta) - T\sin(\alpha + \beta) < 0$ so $R = 0$ (and $F = 0$) as the block lifts from ground. Taking unit vectors **i** and **j** in the directions $\rightarrow$ and $\uparrow$

respectively: $\boldsymbol{T_A} = \begin{pmatrix} -T\sin\beta\sin\alpha \\ T\sin\beta\cos\alpha \end{pmatrix}$, $\boldsymbol{T_B} = \begin{pmatrix} -T\cos\beta\cos\alpha \\ T\cos\beta\sin\alpha \end{pmatrix}$, $\boldsymbol{W} = \begin{pmatrix} 0 \\ -T\tan\frac{1}{2}(\alpha+\beta) \end{pmatrix}$, and the resultant force on the block is $\boldsymbol{T_A} + \boldsymbol{T_B} + \boldsymbol{W} = \begin{pmatrix} T\cos(\alpha+\beta) \\ T\sin(\alpha+\beta) - T\tan\frac{1}{2}(\alpha+\beta) \end{pmatrix}$, which is in the direction $\tan^{-1}\left(\dfrac{\sin(\alpha+\beta) - \tan\frac{1}{2}(\alpha+\beta)}{\cos(\alpha+\beta)}\right)$ (relative to **i**). Since $\alpha + \beta = 2\phi$, this is the direction

$$\tan^{-1}\left(\frac{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta) - \dfrac{\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha+\beta)}}{\cos(\alpha+\beta)}\right) = \tan^{-1}\left(\frac{\sin\frac{1}{2}(\alpha+\beta)\left\{2\cos^2\frac{1}{2}(\alpha+\beta) - 1\right\}}{\cos(\alpha+\beta).\cos\frac{1}{2}(\alpha+\beta)}\right)$$

$= \tan^{-1}\left(\tan\frac{1}{2}(\alpha+\beta)\right) = \frac{1}{2}(\alpha+\beta) = \phi$, noting that $2\cos^2\frac{1}{2}(\alpha+\beta) - 1 = \cos(\alpha+\beta)$.

Q12 As with all such probability questions, there are many ways to approach the problem. The one shown here for part (i) is one that generalises well to later parts of the question. Suppose the container has 3R, 3B, 3G tablets. Then the probability is $\dfrac{3}{9}\times\dfrac{3}{8}\times\dfrac{3}{7} = \dfrac{3}{56}$ for one specified order (e.g. RBG). We then multiply by 3! = 6 for the number of permutations of the 3 colours to get $\dfrac{9}{28}$. The final part of (i) is really a test of whether you realiuse that this is the same situation viewed "in reverse", so the answer is the same.

Using the method above, with a suitable notation, in part (ii) we have

$$\mathrm{P}_3(n) = \frac{n}{3n}\times\frac{n}{3n-1}\times\frac{n}{3n-2}\times 3! = \frac{2n^2}{(3n-1)(3n-2)} \text{ or } \frac{n^3}{\binom{3n}{3}}.$$

Then in (iii), P(correct tablet on each of the n days) = $\mathrm{P}_2(n)\times\mathrm{P}_2(n-1)\times\mathrm{P}_2(n-2)\times \dots \mathrm{P}_2(2)\times\mathrm{P}_2(1)$

$$= \left[\frac{n^2.2!}{2n(2n-1)}\right]\times\left[\frac{(n-1)^2.2!}{(2n-2)(2n-3)}\right]\times\left[\frac{(n-2)^2.2!}{(2n-4)(2n-5)}\right]\times\dots\times\left[\frac{2^2.2!}{4.3}\right]\times\left[\frac{1^2.2!}{2.1}\right]$$

$= \dfrac{(n!)^2 2^n}{(2n)!}$ or $\dfrac{2^n}{\binom{2n}{n}}$. Then, using $n! \approx \sqrt{2n\pi}\left(\dfrac{n}{\mathrm{e}}\right)^n$ and $(2n)! \approx \sqrt{4n\pi}\left(\dfrac{2n}{\mathrm{e}}\right)^{2n}$, we have

$$\text{Prob.} = \frac{2n\pi\times\dfrac{n^{2n}}{\mathrm{e}^{2n}}\times 2^n}{\sqrt{4n\pi}\times\dfrac{2^{2n}n^{2n}}{\mathrm{e}^{2n}}} = \frac{2n\pi\times 2^n}{2\sqrt{n\pi}\times 2^{2n}} = \frac{\sqrt{n\pi}}{2^n}.$$

Q13 First, note that $x \in \{0, 1, 2, 3\}$, so there are four probabilities to work out (actually, three, and the fourth follows by subtraction from 1). The easier ones to calculate are $X = 0$ and $X = 3$, so let us find these first.

For P(X = 0): the 7 pairs from which a singleton can be chosen can be done in ${}^{26}\mathrm{C}_7$ ways; then, we can choose one from each pair in 2^7 ways; so that P(X = 0) = $\dfrac{{}^{26}\mathrm{C}_7\times 2^7}{{}^{52}\mathrm{C}_7}$.

Now $^{26}C_7 = \dfrac{26.25.24.23.22.21.20}{7.6.5.4.3.2.1}$ & $^{52}C_7 = \dfrac{52.51.50.49.48.47.46}{7.6.5.4.3.2.1} \Rightarrow P(X=0) = \dfrac{3520}{5593}$.

P(X = 3): the 3 pairs from 26 can be chosen in $^{26}C_3$ ways; then the one singleton from the remaining 23 pairs can be chosen in $^{23}C_1 = 23$ ways, and the one from that pair in 2^1 ways; so that $P(X=3) = \dfrac{^{26}C_3 \times {}^{23}C_1 \times 2^1}{^{52}C_7} = \dfrac{5}{5593}$ (similarly for calculation).

For P(X = 1): the 1 pair can be chosen in $^{26}C_1 = 26$ ways; the 5 pairs from which a singleton is chosen can be done in $^{25}C_5$ ways; and the singletons from those pairs in 2^5 ways; so that

$P(X=1) = \dfrac{26 \times {}^{25}C_5 \times 2^5}{^{52}C_7} = \dfrac{1848}{5593}$.

For P(X = 2): the 2 pairs can be chosen in $^{26}C_2$ ways; then the 3 pairs from which a singleton is chosen can be done in $^{24}C_3$ ways; and the singletons from those pairs can be chosen in 2^3 ways; so that $P(X=2) = \dfrac{^{26}C_2 \times {}^{24}C_3 \times 2^3}{^{52}C_7} = \dfrac{220}{5593}$.

Then $E(X) = \sum x\,.\,p(x) = \dfrac{1}{5593}\left(0 \times 3520 + 1 \times 1848 + 2 \times 220 + 3 \times 5\right)$

$$= \frac{2303}{5593} = \frac{7 \times 7 \times 47}{7 \times 17 \times 47} = \frac{7}{17}.$$

STEP 2 2013 Hints and Solutions

Question 1.

The gradient of a line from a general point on the curve to the origin can be calculated easily and the gradient of the curve at a general point can be found by differentiation. Setting these two things to be equal will then lead to the correct value of m. A similar consideration of gradients to the origin will establish the second result and if the line intersects the curve twice then a sketch will illustrate that there must be one intersection on each side of the point of contact found in the first case. A similar process will establish the result for part (ii).

For part (iii) the gradient of the line must be smaller than the gradient of the line through the origin which touches the curve, so the intersection with the y-axis must be at a positive value. This means that the conditions of part (ii) are met, which allows for the comparison between π^e and e^π to be made.

The condition given in part (iv) is equivalent to stating that the line is parallel to the one found at the very beginning of the question. This implies that the intersection with the y-axis is at a negative value and so an adjustment to the steps taken in part (ii) will establish the required result.

Question 2.

The obvious substitution in the first part leads easily to the required result. It should then be easy to establish the second result by making the integral into the sum of two integrals and noting that taking out a common factor leaves $(1-x)+x$ to be simplified. Integration by parts will lead to the next result after which taking out one of the factors of $(1-x)$ will allow the integral to be split into a difference of two integrals.

The result in part (ii) is most easily proved by induction. It is necessary to fill in the gap in the factorial on the denominator by multiplying both the numerator and denominator by the missing even number. In alternative approaches, it needs to be remembered that the product of the even numbers up to and including $2n$ can be written as $2^n n!$

The final part is a straightforward substitution, although care needs to be taken with the signs. The final result can be obtained using the relationship established in part (i) as none of the reasoning requires n to be an integer.

Question 3.

For it to be possible for the cubic to have three real roots it must have two stationary points. Since the coefficient of x^3 is positive it must have a specific shape. A sketch will show that only the two cases given will result in an intercept with the y-axis at a negative value.

In order for the cubic in part (ii) to have three positive roots, both of the turning points must be at positive values of x. Differentiation will allow most of the results to be established. The condition that $c < 0$ is needed to ensure that the leftmost root is also positive.

The condition $ab < 0$ implies that there must be a turning point at a positive value of x. The shape of the graph is as in part (i), but this time the intersection with the y-axis is at a positive value. This is sufficient to deduce the signs of the roots.

For part (iv) it is easiest to note that changing the value of c does not (as long as c remains negative) change whether or not the conditions of (*) are met. As this represents a vertical translation of the graph any example of a case satisfying (*) can be used to create an answer for this part by translating the graph sufficiently far downwards.

Question 4.

The equations of the line and circle are easily found and so the second point of intersection (and so the coordinates of M) can be easily found. The two parts of this question then involve regarding the coordinates of M as parametric equations.

In part (i) a is the parameter and is restricted so that the point that the line passes through is inside the circle. This gives a straight line between the points which result from the cases $a = -1$ and $a = 1$. The length of this line can be determined easily from the coordinates of its endpoints.

In part (ii) it is again quite easy to eliminate the parameter from the pair of equations and the shapes of the loci should be easily recognised. In part (b) however, the restriction on the values of b need to be considered as the locus is not the whole shape that would be identified from the equation.

Question5.

Simple applications of the chain rule lead to relationships that will allow the three cases of zero gradients to be identified in part (i).

In part (ii) the relationships follow easily from substitution and therefore the three stationary points identified in part (i) must all exist. By considering the denominator there are clearly two vertical asymptotes and the numerator is clearly always positive. Additionally, the numerator is much larger than the denominator for large values of x. Given this information there is only one possible shape for the graph.

In part (iii) the solutions of the first equation will already have been discovered when the coordinates of the stationary points in part (ii) were calculated. The range of values satisfying the first inequality should therefore be straightforward. One of the solutions of the second equation should be easy to spot, and consideration of the graph shows that there must be a total of six roots. Applying the two relationships about the values of f will allow these other roots to be found. The solution set for the inequality then follows easily from consideration of the graph.

Question 6.

The definition of the sequence can be used to find a relationship between u_{n+2} and u_n and therefore also a relationship between u_n and u_{n-2}. Taking the difference of these then leads to the required result.

It is clear from the definition of the sequence that, if one term is between 1 and 2, then the next term will also be between 1 and 2. This is then easy to present in the form of a proof by induction for part (ii).

The result of part (i) shows that the sequence in part (iii) is increasing and the result proved in part (ii) shows that it is bounded above. The theorem provided at the start of the question therefore shows that the sequence converges. Similarly the second sequence is bounded below and decreasing (and therefore if the terms are all multiplied by -1 a sequence will be generated which is bounded above and increasing). Therefore the second sequence also converges to a limit.

The relationship between u_n and u_{n-2} established in part (i) can then be used to find the value of this limit and, as it is the same for both the odd terms and the even terms, the sequence must tend to the same limit as well.

Finally, starting the sequence at 3 will still lead to the same conclusion as the next term will be between 1 and 2 and all further terms will also be within that range, so all of the arguments will still hold for this new sequence.

Question 7.

A solution of the equation should be easy to spot and a simple substitution will establish the new solution that can be generated from an existing one. This therefore allows two further solutions to be found easily by repeated application of this result.

In part (ii) write $x = 2m + 1$ and $y = 2n$ and then substitute into (*). With some simplification the required relationship will be established.

Since b is a prime number there is only two ways in which it can be split into a product of two numbers ($1 \times b^3$ and $b \times b^2$). The right hand side of the equation is clearly a difference of two squares and therefore a pair of simultaneous equations can be solved to give expressions for a and c^2. Finally, the expression for c^2 is similar to the relationship established in part (ii), so solutions to the original equation can be used to generate values of a, b and c which satisfy this equation.

Question 8.

Begin by calculating the largest area of a rectangle with a given width and then maximize this function as the width of the rectangle is varied. The definition of x_0 can be reached by setting the derivative of the area function to 0.

The definition of g involves the differentiation of an integral of f which uses the variable t as the upper limit. The derivative of $tg(t)$ is therefore $f(t)$. The next statement relates the area bounded by the curve and the line $y = f(t)$ with the area of the largest rectangle with edges parallel to the axes that can fit into that space, so the first area must be greater and since that integral is equal to $tg(t) - tf(t)$ the result that follows is easily deduced.

The final part of the question involves finding expressions for $A_0(t)$ and $g(t)$ and then simplifying the relationship established at the end of part (ii).

Question 9.

Resolving the forces vertically will establish the first result. For the second part of the question it can be established that all of the frictional forces are equal in magnitude by taking moments about the centre of one of the discs. Resolving forces vertically and horizontally for the discs individually will then lead to simultaneous equations that can be solved for the magnitudes of the reaction and frictional forces.

Since the discs cannot overlap there is a minimum value that θ can take and the value of $\frac{\sin\theta}{1+\cos\theta}$ is increasing as θ increases. This allows the smallest possible value of the frictional force between the discs to be calculated and therefore it can be deduced that no equilibrium is possible if the coefficient of friction is below this minimum value.

Question 10.

Following the usual methods of considering horizontal and vertical parts of the motion will lead to the first result (some additional variables will need to be used, but they will cancel out to reach the final result.

If B and C are the same point then the result in part (i) can be applied for this point which will give an equation which is easily solved to give $\alpha = 60°$ once the double angle formula has been applied.

For the final part it is possible to find the times at which the particle reaches each of the two points. The two equations reached can then be used to find an expression for the difference between the time at which the particle reaches each of the two points and then it can easily be deduced whether this is positive or negative, which will show which point is reached first.

Question 11.

The standard methods of conservation of momentum and the law of restitution will allow the speeds after the second collision to be deduced. A third collision would have to be between the first and second particles and this will only happen if the velocity of the first particle is greater than that of the second one.

Providing a good notation is chosen to avoid too much confusion, it is possible to find the velocities after the third collision and then consider the velocities of the second and third particles to determine whether or not there is a fourth collision.

Question 12.

The formula for the expectation of a random variable should be well known and both of the expectations can easily be written in terms of α and β.

Similarly, the formula for variance should be well known and so it is a matter of rearranging the sums in such a way as to reach the forms given in the question. Note that the definitions of α and β are such that $e^{\lambda} = \alpha + \beta$.

Since the $Var(X+Y) = Var(U)$ the equation in the final part of the question can be rewritten in terms of the variables defined at the start of the question. It can then be shown that this is not possible for any non-zero value of λ.

Question 13.

An alternating run of length 1 must be two results showing the same side of the coin. It is then easy to show that the probability is as given. Similarly a straight run of length 1 must be two different results (in either order) and so the probability can again be calculated easily. The terms involved are those in the expansion of $(p \pm q)^2$ and so starting with the statement that $(p-q)^2 \geq 0$ then relationship between the two probabilities can be established.

An alternating run of length 2 must be one result followed by the other one twice, while a straight run of length 2 must be two identical results followed by the other one. They will therefore be calculated by the same sums (with the products in a different order each time) so the probabilities must be equal. By considering the ways in which runs of length 3 can be obtained it is clear that these two probabilities must also be equal.

An alternating run of length $2n$ must be n of each of the two possibilities followed by a repeat of whichever came last. A straight run of length $2n$ must be $2n$ of one of the possibilities followed by 1 of the other. Taking the difference between these two probabilities gives an expression which can be seen to always have the same sign, which will determine which probability is greater. A similar method will also work for the final case.

STEP 3 2013 Hints and Solutions

1. The first two results, whilst not necessarily included in current A2 specifications, are standard work. Applying them, $\int_0^{\frac{1}{2}\pi} \frac{1}{1+a\sin x} \; dx = 2\int_0^1 \frac{1}{(1-a^2)+(t+a)^2} dt$, which can then be evaluated using a change of variable to give $\frac{2}{\sqrt{1-a^2}}\left(\tan^{-1}\frac{1+a}{\sqrt{1-a^2}} - \tan^{-1}\frac{a}{\sqrt{1-a^2}}\right)$. To simplify this to obtain the required result, $\tan\left(\tan^{-1}\frac{1+a}{\sqrt{1-a^2}} - \tan^{-1}\frac{a}{\sqrt{1-a^2}}\right)$ must be simplified using the relevant compound angle formula.

It is fairly straightforward to show that $I_{n+1} + 2I_n = \int_0^{\frac{1}{2}\pi} \sin^n x \, dx$, so applying this for $n = 2, 1, 0$ and applying the main result of the question to evaluate I_0 , gives $I_3 = \left(\frac{9}{4} - \frac{8\sqrt{3}}{9}\right)\pi - 2$

2. It is elegant to multiply by the denominator, then differentiate implicitly, and finally multiply by the same factor again to achieve the desired first result. The general result can be proved by then using induction, or by Leibnitz, if known. The general result can be used alongside the expression for y, and the first derived result with the substitution $x = 0$ to show that the general term of the Maclaurin series for even powers of x is zero, and for odd powers of x is $\frac{2^{2r}(r!)^2}{(2r+1)!} x^{2r+1}$. Thus, as $y = x + \frac{2^2}{3!}x^3 + \frac{4^2 2^2}{5!}x^5 + \cdots$ the required infinite sum is $\frac{y}{x}$ with $x = \frac{1}{2}$, that is $\frac{2\pi\sqrt{3}}{9}$.

3. The scalar product of p_i with $\sum p_r$, which is of course zero, can be expanded giving $p_i.p_i = 1$ and three products $p_i.p_j$ which are equal by symmetry, giving the required result. Expanding the expression suggested in (i), gives $\sum_{i=1}^4 (p_i.p_i - 2x.p_i + x.x)$, which, bearing in mind that $p_i.p_i = 1$, $x.x = 1$, and that $x.\sum_{i=1}^4 p_i = 0$, gives the correct result. Considering that $p_1.p_2 = -\frac{1}{3}$, $p_2.p_2 = 1$, and that a is positive, enables the given values to be found. Similarly $p_1.p_3 = -\frac{1}{3}$, $p_2.p_3 = -\frac{1}{3}$, and $p_3.p_3 = 1$ yields P_3, $P_4 = \left(-\frac{\sqrt{2}}{3}, \pm\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{3}\right)$. In (iii), using the logic of (i), $(XP_i)^4 = \left((p_i - x).(p_i - x)\right)^2 = 4(1 - x.p_i)^2$, as required. Expanding this, and using the coordinates of X and those of P_i that have been found,

$$\sum_{i=1}^4 (XP_i)^4 = 16 + 4\left(z^2 + \left(\frac{2\sqrt{2}}{3}x - \frac{1}{3}z\right)^2 + \left(-\frac{\sqrt{2}}{3}x + \frac{\sqrt{2}}{\sqrt{3}}y - \frac{1}{3}z\right)^2 + \left(-\frac{\sqrt{2}}{3}x - \frac{\sqrt{2}}{\sqrt{3}}y - \frac{1}{3}z\right)^2\right)$$

$= 16 + 4\left(\frac{4}{3}x^2 + \frac{4}{3}y^2 + \frac{4}{3}z^2\right) = \frac{64}{3}$ which is sufficient.

4. The initial result is obtained by expanding the brackets and expressing the exponentials in trigonometric form. The (2n)th roots of -1 are $e^{i\frac{2m+1}{2n}\pi}$, $-n \le m \le n-1$, which lead to the factors of $z^{2n} + 1$ and these paired using the initial result give the required result. Part (i) follows directly from substituting $z = i$ in the previous result, and as n is even, $z^{2n} + 1 = 2$. Using the given factorisation in part (ii), the general result can be simplified by the factor

$z^2 - 2z\cos\frac{n}{2n}\pi + 1 = z^2 + 1$. Again substituting $z = i$, and that $\cos\frac{2n-r}{2n}\pi = -\cos\frac{r}{2n}\pi$ gives the evaluation required.

5. Writing q^nN as $qq^{n-1}N$, and employing the permitted assumption, as p and q are coprime, p divides $q^{n-1}N$. Repetitions of this argument imply finally that p divides N . Letting $N = pQ_1$, $q^nQ_1 = p^{n-1}$. Continuing this argument similarly gives the result $N = kp^n$. As a consequence, $q^nk = 1$, and thus q and k must both be 1 . Thus if $\sqrt[n]{N} = \frac{p}{q}$ where p and q are coprime, it is rational and can be written in lowest terms, then $q^nN = p^n$ and so $q = 1$ and thus $\sqrt[n]{N}$ is an integer. Otherwise, $\sqrt[n]{N}$ cannot be written as $\frac{p}{q}$, that is, it is irrational.

For (ii), using the same logic as in part (i), as b^a divides a^ad^b , b^a divides d^b, so $d^b = kb^a$, for some k . Likewise, $a^a = k'c^b$, for some integer k' , and thus $k'k = 1$, so $k = k' = 1$, and $d^b = b^a$. If p is a prime factor of d , then p divides d^b , and so b^a too. Writing $b^a = bb^{a-1}$, using the logic of the very first part of the question, if p does not divide b , p divides b^{a-1}, and repetition of this argument leads to a contradiction. So p is a prime factor of b . p^{mb} and p^{na} is the highest power of p that divides $d^b = b^a$. So $mb = na$, and $b = \frac{na}{m}$. So p^n divides na , but as a and b are coprime, p^n divides n and thus $p^n \leq n$. By the given result, this means $p = 1$, and as b is only divisible by 1 , $b = 1$. If r is a positive rational $\frac{a}{b}$, such that $r^r = \frac{c}{d}$ is rational, then $a^ad^b = b^ac^b$ so $b = 1$ and r is a positive integer.

6. The opening result is the triangle inequality applied to OW, OZ, and WZ where OW and OZ are represented by the complex numbers w and z.
Part (i) relies on using $|z - w|^2 = (z - w)(z - w)^*$, $(z - w)^* = (z^* - w^*)$, $|zw| = |z|\,|w|$, and substituting $wz^* + zw^* = (E - 2|zw|)$. Having obtained the desired equation , the reality of E is apparent from the reality of the other terms and its non-negativity is obtained from the opening result of the question. Part (ii) relies on the same principles as part (i).
The inequality can be most easily obtained by squaring it, and substituting for both numerator and denominator on the left hand side using parts (i) and (ii), and algebraic rearrangement leads to $E(1 - |z|^2)(1 - |w|^2) \geq 0$ which is certainly true. The argument is fully reversible as $|z| > 1$, and $|w| > 1$, $|zw^*| > 1$, and so $1 - zw^* \neq 0$ so the division is permissible, and the square rooting of the inequality causes no problem as the quantities are positive. The working follows identically if $|z| < 1$, and $|w| < 1$.

7. As $\frac{dE}{dx} = 2\,\frac{dy}{dx}\left(\frac{d^2y}{dx^2} + y^3\right)$ is zero for all x , $E(x)$ is constant, and $E(x) = \frac{1}{2}$ using the initial conditions. The deduction follows from the non-negativity of $\left(\frac{dy}{dx}\right)^2$. In part (ii), it can be shown that $\frac{dE}{dx} = -2x\left(\frac{dv}{dx}\right)^2 \leq 0$ for $x \geq 0$, and as initially $E(x) = \frac{10}{3}$, the deduction for $\cosh v(x)$ follows in the same way as that in part (i). In part (iii), the choice of $E(x)$ relies on $2\int(w\cosh w + 2\sinh w)dw$ so $E(x) = \left(\frac{dw}{dx}\right)^2 + 2\,(w\sinh w + \cosh w)$. Then $\frac{dE}{dx} = -2\left(\frac{dw}{dx}\right)^2(5\cosh x - 4\sinh x - 3) = -2\left(\frac{dw}{dx}\right)^2\frac{e^{-x}}{2}\,(e^x - 3)^2$, and initially $E(x) = \frac{5}{2}$.
The final result can be deduced as in the previous parts, with the additional consideration that $w\sinh w \geq 0$.

8. The sum is evaluated by recognising that it is a geometric progression with common ratio $e^{2i\pi/n}$ which may be summed using the standard formula and as $1 - e^{2i\pi/n} \neq 0$, the denominator

is non-zero so the sum is zero. By simple trigonometry, $s = d - r\cos\theta$. As $r = ks$, $r = \frac{kd}{1+k\cos\theta}$. Thus $l_j = \frac{kd}{1+k\cos\theta} + \frac{kd}{1+k\cos(\theta+\pi)}$ where $\theta = \alpha + (j-1)\pi/n$. Simplifying, $l_j = \frac{2kd}{1-k^2\cos^2\theta}$. The summation of the reciprocals of this expression is simply found using a double angle formula and then by expressing the trigonometric terms as the real part of the sum at the start of the question.

9. The volume is obtained as a volume of revolution $V = \int_x^R \pi(R^2 - t^2)dt$ which gives the result. Similarly, Newton's 2nd law gives $\frac{4}{3}\pi R^3 \rho_s \ddot{x} = V\rho g - \frac{4}{3}\pi R^3 \rho_s g$ which simplifies to the required result. Substituting $x = \frac{1}{2}R$ when $\ddot{x} = 0$ gives $\rho_s = \frac{5}{32}\rho$. Substituting $x = \frac{1}{2}R + y$ yields $\frac{5}{8}R^3\ddot{y} = g\left(-\frac{9}{4}R^2y + \frac{3}{2}Ry^2 + y^3\right)$, so for small y this approximates to SHM with period $\frac{\pi}{3}\sqrt{\frac{10R}{g}}$.

10. The initial result can be obtained in a number of different ways, but probably use of the parallel axes rule is the simplest. Conserving angular momentum about P,

$mu(a+x) = mv(a+x) + \frac{1}{3}M(a^2+3x^2)\omega$ where v is the velocity of the particle after impact, and ω is the angular velocity of the beam after the impact, and by Newton's experimental law of impact $(a+x)\omega - v = eu$. **Eliminating** v between these two equations gives the quoted expression for ω. Substituting $m = 2M$, for maximum ω, $\frac{d\omega}{dx} = 0$. This gives a quadratic equation, with solutions $x = -\frac{1}{3}a$ and $x = -\frac{5}{3}a$. The latter is not feasible and the former can be shown to generate a maximum which equates to the given result.

11. As the distance from the vertex to the centre of the equilateral triangle is a, the extended length of each spring is $\frac{a}{\cos\theta}$ giving the tension in each as $kmg\frac{\left(\frac{a}{\cos\theta}-a\right)}{a}$ which simplifies to the given result. Resolving vertically $3T\sin\theta = 3mg$, and using the result for T, substituting $\theta = \frac{\pi}{6}$, and rationalising the denominator gives the required value for k. Conserving energy, when $\theta = \frac{\pi}{3}$, gravitational potential energy is $-3mga\tan\frac{\pi}{3}$, elastic potential energy is $\frac{3}{2}kmg\frac{\left(\frac{a}{\cos\frac{\pi}{3}}-a\right)^2}{a} = \frac{3}{2}kmga\left(\frac{1}{\cos\frac{\pi}{3}}-1\right)^2$, whereas when $\theta = \frac{\pi}{6}$, gravitational potential energy is $-3mga\tan\frac{\pi}{6}$, elastic potential energy is $\frac{3}{2}kmga\left(\frac{1}{\cos\frac{\pi}{6}}-1\right)^2$, and kinetic energy is $\frac{3}{2}mV^2$ hence giving V^2.

12. $P(X_1 = 1) = \frac{a}{n}$, so $E(X_1) = \frac{a}{n}$. There are $\frac{n!}{a!b!}$ arrangements of the As and Bs, and the number of arrangements with a B in the $(k-1)$ th place and an A in the k th place is $\frac{(n-2)!}{(a-1)!(b-1)!}$, so $P(X_k = 1) = \frac{ab}{n(n-1)}$ for $2 \le k \le n$, and $E(X_i) = \frac{ab}{n(n-1)}$ if $i \ne 1$. These combine to give $E(S)$ correctly.

$X_1X_j = 1$ only if the first letter is an A, the $(j-1)$ th letter is a B, and the j th letter is an A. This has probability $\frac{(n-3)!}{(a-2)!(b-1)!}/\frac{n!}{a!b!}$ giving $E(X_1X_j)$ correctly.

$X_iX_j = 1$ only if the $(i-1)$ th letter is a B, and the i th letter is an A, the $(j-1)$ th letter is a B, and the j th letter is an A which has probability $\frac{(n-4)!}{(a-2)!(b-2)!}/\frac{n!}{a!b!}$ so $E(X_iX_j) = \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$, and thus $\sum_{j=i+2}^{n} E(X_iX_j) = (n-i-1)\frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$ and so

$\sum_{i=2}^{n-2}\left(\sum_{j=i+2}^{n} E(X_iX_j)\right) = \sum_{i=2}^{n-2}\left((n-i-1)\frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}\right)$ which yields the required result.

$S^2 = \sum_{i=1}^{n} {X_i}^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 2X_i X_j$ so $E(S^2) = \frac{a(b+1)}{n} + \frac{a(a-1)b(b+1)}{n(n-1)}$ which can be used to obtain $Var(S)$ correctly.

13. integrating $0 \le f(t) \le M$ between limits of 0 and x gives the result of (a) (i), and integrating the left hand side by parts yields part (ii). As $kF(y)f(y)$ is a probability density function, $\int_0^1 k\,F(y)f(y)dy = 1$, which can be evaluated using the result of (a) (ii) with $2g(x) = k$ and so $k = 2$. $E(Y^n) = \int_0^1 y^n\,2F(y)f(y)dy \le \int_0^1 y^n\,2Myf(y)dy = 2M\mu_{n+1}$ and using (a) (ii) , $E(Y^n) = \int_0^1 y^n\,2F(y)f(y)dy = 1 - n\int_0^1 y^{n-1}\left(F(y)\right)^2 dy$, as $\int_0^1 y^{n-1}\left(F(y)\right)^2 dy \le \int_0^1 y^{n-1}My\,F(y)dy = M\int_0^1 y^n\,F(y)dy$, integration by parts gives $\int_0^1 y^n\,F(y)dy = \frac{1}{n+1} - \frac{1}{n+1}\mu_{n+1}$. Part (iii) is derived from part (ii) by rearranging

$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \le 2M\mu_{n+1}$ and making μ_{n+1} the subject, then translating $n+1$ to n .

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The Admissions Testing Service

Cambridge English
Language Assessment
1 Hills Road
Cambridge
CB1 2EU
United Kingdom
Tel: +44 (0)1223 553366
Email: admissionstests@cambridgeassessment.org.uk

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Language Assessment
Part of the University of Cambridge